INTEGRATING RATIONAL FUNCTIONS

To find $\int \frac{N(x)}{D(x)}dx$**, where** $N(x)$ **and** $D(x)$ **are both polynomials, and no cancellation is possible**

Some rational integrands require partial fractions decomposition, others do not, and still others are a combination of the two. The process below outlines when you should and should not use partial fractions decomposition.

If degree of $N(x) \geq$ degree of $D(x)$ (ie. numerator has same degree as, or higher degree than, denominator)

1. Perform polynomial long division
2. Rewrite integrand as polynomial + remainder (with degree < degree of $D(x)$)
3. Use processes below to find the integral of $\frac{remainder}{D(x)}$ (ie. set $N(x) = remainder$ and continue)

If degree of $D(x) = 1$ (ie. denominator is linear)

1. Let $u = D(x)$ & perform $u-$substitution

   (or use guess & check – antiderivative is a multiple of $\ln|D(x)|$)

eg.

$$\int \frac{13}{7x-5}dx \qquad u = 7x-5 \;\Rightarrow\; \frac{du}{dx} = 7 \;\Rightarrow\; dx = \frac{1}{7}du$$

$$= \int \frac{13}{7}\frac{du}{u} = \frac{13}{7}\ln|u| + C = \frac{13}{7}\ln|7x-5| + C$$

If $N(x) = k \cdot D'(x)$ (ie. numerator is constant multiple of derivative of denominator) **TYPE 1**

1. Let $u = D(x)$ & perform $u-$substitution

   (or use guess & check – antiderivative is a multiple of $\ln|D(x)|$)

eg.

$$\int \frac{20-5x}{x^2-8x+25}dx \qquad u = x^2-8x+25 \;\Rightarrow\; \frac{du}{dx} = 2x-8 \;\Rightarrow\; (20-5x)dx = -\frac{5}{2}(2x-8)dx = -\frac{5}{2}du$$

$$= \int -\frac{5}{2}\frac{du}{u} = -\frac{5}{2}\ln|u| + C = -\frac{5}{2}\ln|x^2-8x+25| + C$$

If degree of $D(x) = 2$ and is irreducible
(ie. denominator is quadratic with negative discriminant, so denominator has no real roots / only complex roots)

If $N(x) = c$ (ie. numerator is constant) **TYPE 2**

1. Factor leading coefficient from $D(x)$ (ie. so denominator starts with x^2)
2. Complete the square for $D(x) = (x+h)^2 + a^2$
3. Factor a^2 from denominator, let $u = \frac{x+h}{a}$ & perform $u-$substitution

   (or use $\int \frac{1}{(x+h)^2 + a^2}dx = \frac{1}{a}\tan^{-1}\frac{x+h}{a}$)

eg.

$$\int \frac{7}{2x^2-16x+50}dx = \frac{7}{2}\int \frac{1}{x^2-8x+25}dx = \frac{7}{2}\int \frac{1}{(x-4)^2+3^2}dx = \frac{7}{2}\frac{1}{3}\tan^{-1}\frac{x-4}{3} + C = \frac{7}{6}\tan^{-1}\frac{x-4}{3} + C$$

If $N(x) = ax + b$ (ie. numerator is linear) **TYPE 3**

1. Use technique similar to partial fractions shortcut to rewrite numerator as $A \cdot D'(x) + B \cdot a$
   (ie. constant multiple of derivative of denominator +
   constant multiple of square root of "leftover" constant from completing square in denominator)
   (see Partial Fractions Decomposition handout (special note regarding Math 1B))
2. Split integrand into integrand of **TYPE 1** + integrand of **TYPE 2**
3. Use processes above (NOTE: no absolute values required in $\ln(D(x))$)

eg.

$$\int \frac{3x-17}{x^2-8x+25}\,dx = \int \frac{A(2x-8)+B}{(x-4)^2+3^2}\,dx$$

$x = 4:$	$3(4)-17 = A(0)+B$	$\Rightarrow B = -5$
coefficient of $x:$	$3 = 2A$	$\Rightarrow A = \frac{3}{2}$

$$= \int \frac{\frac{3}{2}(2x-8)-5}{(x-4)^2+3^2}\,dx = \frac{3}{2}\int \frac{2x-8}{x^2-8x+25}\,dx - 5\int \frac{1}{(x-4)^2+3^2}\,dx$$

$$= \frac{3}{2}\ln(x^2-8x+25) - \frac{5}{3}\tan^{-1}\frac{x-4}{3} + C$$

NOTE: **TYPE 3** rational functions can also be integrated using a trigonometric substitution, usually requiring more work

eg. $$\int \frac{3x-17}{x^2-8x+25}\,dx = \int \frac{3x-17}{(x-4)^2+9}\,dx$$ Let $x = 4+3\tan\theta$, so $dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{3(4+3\tan\theta)-17}{(3\tan\theta)^2+9}3\sec^2\theta\,d\theta = \int \frac{9\tan\theta-5}{3}\,d\theta = 3\ln|\sec\theta| - \frac{5}{3}\theta + C$$

$$= 3\ln\sqrt{(x-4)^2+9} - \frac{5}{3}\tan^{-1}\frac{x-4}{3} + C = \frac{3}{2}\ln((x-4)^2+9) - \frac{5}{3}\tan^{-1}\frac{x-4}{3} + C$$

All other cases require partial fractions

1. Perform partial fractions decomposition (see Partial Fractions Composition handout)
   NOTE: for irreducible quadratic denominators $d(x) = ax^2 + bx + c$
   or powers of these factors, ie. $[d(x)]^n$ or $(ax^2+bx+c)^n$
   write numerator in $A \cdot d'(x) + B$ form for **TYPE 3** to save work later on
2. For all resulting partial fractions with linear and irreducible quadratic denominators:
   Integrate using processes above
3. For all resulting partial fractions with denominator $[d(x)]^n = (ax+b)^n$ (ie. power of linear factor):
   Let $u = ax + b$ & perform $u-$substitution to integrate
   (or use guess & check – antiderivative is a multiple of $\frac{1}{(ax+b)^{n-1}}$)
4. For all partial fractions with denominator $[d(x)]^n = (ax^2+bx+c)^n$ (ie. power of irreducible quadratic factor):
   Split integrand into integrand with numerator $A \cdot d'(x)$ + integrand with numerator B
   For first integrand:
   - Let $u = ax^2 + bx + c$ & perform $u-$substitution
   - (or use guess & check – antiderivative is a multiple of $\frac{1}{(ax^2+bx+c)^{n-1}}$)

   For second integrand:
   - Factor leading coefficient from $ax^2 + bx + c$ (ie. so irreducible quadratic starts with x^2)
   - Complete the square for $x^2 + Bx + C = (x+h)^2 + k^2$
   - Let $x + h = k\tan\theta$ & perform trigonometric substitution
   - NOTE: This is the hardest type – there will be no required problems of this type on tests

Practice against the following examples:

[A] $\int \frac{7}{3x+8}\,dx$

[B] $\int \frac{6-9x}{3x^2-4x-4}\,dx$

[C] $\int \frac{5}{4x^2+24x+52}\,dx$

[D] $\int \frac{5x-7}{x^2+8x+25}\,dx$

[E] $\int \frac{-20x-12}{(x+1)^2(x^2+4x+7)}\,dx$

[F] $\int \frac{3x^3-11x^2-49x}{x^2-5x-6}\,dx$

Major steps in solutions

[A] $\int \frac{7}{3x+8}\,dx = \frac{7}{3}\ln|3x+8|+C$

[B] **TYPE 1** $\int \frac{6-9x}{3x^2-4x-4}\,dx = -\frac{3}{2}\ln|3x^2-4x-4|+C$

[C] **TYPE 2** $\int \frac{5}{4x^2+24x+52}\,dx = \frac{5}{4}\int \frac{1}{x^2+6x+13}\,dx = \frac{5}{4}\int \frac{1}{(x+3)^2+2^2}\,dx = \frac{5}{8}\tan^{-1}\frac{x+3}{2}+C$

[D] **TYPE 3** $\int \frac{5x-7}{x^2+8x+25}\,dx = \int \frac{\frac{5}{2}(2x+8)-27}{(x+4)^2+3^2}\,dx = \frac{5}{2}\int \frac{2x+8}{(x+4)^2+3^2}\,dx - 27\int \frac{1}{(x+4)^2+3^2}\,dx$

$= \frac{5}{2}\ln(x^2+8x+25) - 9\tan^{-1}\frac{x+4}{3}+C$

[E] **MIXED** $\int \frac{-20x-12}{(x+1)^2(x^2+4x+7)}\,dx = \int\left(\frac{-6}{x+1}+\frac{2}{(x+1)^2}+\frac{3(2x+4)+4}{(x+2)^2+(\sqrt{3})^2}\right)dx$

$= \int \frac{-6}{x+1}\,dx + \int \frac{2}{(x+1)^2}\,dx + 3\int \frac{2x+4}{x^2+4x+7}\,dx + 4\int \frac{1}{(x+2)^2+(\sqrt{3})^2}\,dx$

$= -6\ln|x+1| - \frac{2}{x+1} + 3\ln(x^2+4x+7) + \frac{4}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}+C$

[F] **MIXED** $\int \frac{3x^3-11x^2-49x}{x^2-5x-6}\,dx = \int\left(3x+4+\frac{-11x+24}{(x-6)(x+1)}\right)dx = \int\left(3x+4-\frac{6}{x-6}-\frac{5}{x+1}\right)dx$

$= \frac{3}{2}x^2+4x-6\ln|x-6|-5\ln|x+1|+C$